ON CERTAIN CHAINS OF THEOREMS IN REFLEXIVE GEOMETRY

A DISSERTATION

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

FLORA DOBLER SUTTON

BALTIMORE
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ON CERTAIN CHAINS OF THEOREMS IN REFLEXIVE GEOMETRY.

BY FLORA D. SUTTON.

§ 1. INTRODUCTION.

It is of interest to extend properties of the triangle to more than three lines and, when possible, to n lines of a plane. This can often be done by a process called geometrical interpolation. As an illustration, I consider, in what follows, a problem proposed by Desboves in his "Questions de Géométrie Élementaire," * namely: "Des trois sommets a, b, c d'un triangle on abaisse des perpendiculaires ap, bq, cr sur une droite quelconque de son plan, pius, des points p, q, r des perpendiculaires sur bc, ac, ab: ces trois dernières droites se coupent en un même point." A discussion of this problem, by means of trilinear coördinates, is given by Kazimierz Cwojdzinski in *Archiv der Mathematik und Physik*.†

In the present paper some extensions of this theorem are considered as indicated in the following scheme:

Undirected Lines.

First Chain.		Second Chain.	
1—a.	3 lines	2—a.	4 lines
1—b.	4 lines	2—b.	5 lines
1—c.	5 lines	2—c.	6 lines
1—g.	general statement	2—g.	general statement

Directed Lines.

I—a.	4 lines	II—a.	5 lines
I—b.	5 lines	II—b.	6 lines
I—g.	general statement	II—g.	general statement

We name a point of the plane by a complex number.

§ 2. SOME FUNDAMENTAL FORMULÆ IN REFLEXIVE GEOMETRY.

1. *Reflexion in a Line.*

If the two triangles x, a, b and y, b, a are inversely similar, and y is the reflexion of x, b the reflexion of a, and a the reflexion of b, in one and the same line, then the two triangles x, a, b and $\bar{y}$, $\bar{b}$, $\bar{a}$ are directly similar.

* 1875; page 241; No. 77.

† 1901; Vol. 1; pp. 175–180.

The condition that any two triangles x_1, x_2, x_3 and y_1, y_2, y_3 be directly similar is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0; \tag{1}$$

therefore, since the two triangles x, a, b and $\bar{y}, \bar{b}, \bar{a}$ are directly similar,

$$\begin{vmatrix} x & a & b \\ \bar{y} & \bar{b} & \bar{a} \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{2}$$

Expanding this determinant, we obtain

$$\frac{x}{(a-b)} + \frac{\bar{y}}{(\bar{a}-\bar{b})} = \frac{a\bar{a} - b\bar{b}}{(a-b)(\bar{a}-\bar{b})}, \tag{3}$$

which can be written more symmetrically, as

$$\frac{x-b}{a-b} + \frac{\bar{y}-\bar{b}}{\bar{a}-\bar{b}} = 1 \tag{4}$$

or

$$\frac{x-a}{a-b} + \frac{\bar{y}-\bar{a}}{\bar{a}-\bar{b}} = 1. \tag{5}$$

If $b = 0 = , \bar{b}$ (4) becomes

$$\frac{x}{a} + \frac{\bar{y}}{\bar{a}} = 1, \tag{6}$$

which is the standard equation for reflexion in a line.

2. *Map-equation of the Double Parabola.**

The map-equation of the double parabola is

$$x = \frac{A_1}{(\alpha_1 - t)^2} + \frac{A_2}{(\alpha_2 - t)^2} \tag{1}$$

(where α_1, α_2, t are turns or orthogonal numbers), provided the cusp condition $dx/dt = 0$ is satisfied for t a turn.

Applying the cusp condition to (1) we have

$$\frac{dx}{dt} = \sum^{2} \frac{2A_i}{(\alpha_i - t)^3} = 0. \tag{2}$$

The conjugate equation of (2) is

$$\sum^{2} \frac{2\bar{A}_i \alpha_i^3 t^3}{(\alpha_i - t)^3} = 0. \tag{$\bar{2}$}$$

* The name is due to Clifford.

Hence, the condition that (1) has a cusp is satisfied when

$$A_i = \overline{A}_i \alpha_i^3. \tag{3}$$

And, in general, the map-equation of an n-fold parabola is

$$x = \sum^{n} \frac{A_i}{(\alpha_i - t)^2}, \tag{4}$$

provided the cusp condition $dx/dt = 0$ is satisfied for a turn t, that is, when (3) is satisfied for $i = 1, 2, 3, \cdots, n$.

3. *The Equation of a Directed Line.*

Directed lines are lines which possess a right- and left-hand side, and therefore should be marked with an arrow head.

The normal form of a line, in rectangular coördinates, is

$$X \cos \alpha + Y \sin \alpha = p, \tag{1}$$

where p is the $\perp$ distance from the line to the origin.

If, now, we employ the circular coördinates $x = X + iY$, $\bar{x} = X - iY$, instead of the rectangular coördinates, (1) becomes

$$(x + \bar{x}) \cos \alpha + \left(\frac{x - \bar{x}}{i}\right) \sin \alpha = 2p \tag{2}$$

or

$$x(\cos \alpha - i \sin \alpha) + \bar{x}(\cos \alpha + i \sin \alpha) = 2p. \tag{3}$$

But $(\cos \alpha - i \sin \alpha)$ is a turn; let us represent it by t, then

$$(\cos \alpha + i \sin \alpha) = 1/t.$$

Putting these values in (3) we have

$$xt + \bar{x}/t = 2p \tag{4}$$

as the equation of a directed line.

4. *Distance from a Straight Line.*

In the case of directed lines, distance = length $\div$ direction.

Let 0 be the reflexion of $a = 2p/t$ (where p is real) in the axis, then the equation of the axis is

$$yt + \bar{x}/t = 2p. \tag{1}$$

From the figure we see that the distance of the point x from the axis is

$$\frac{x - y}{2 \cdot 1/t}. \tag{2}$$

But from (1) we have that

$$ty = -\bar{x}/t + 2p. \tag{3}$$

Therefore,

$$\frac{x - y}{2 \cdot 1/t} = \tfrac{1}{2}(tx + \bar{x}/t - 2p). \tag{4}$$

Thus, twice the distance of a point x from a line is expressed by the equation

$$2D = tx + \bar{x}/t - 2p.$$

Undirected Lines.

§ 3. FIRST CHAIN OF THEOREMS.

1—*a*. Given three lines λ_1, λ_2 and λ_3 tangent to the parabola

$$x = \frac{1}{(1 + t)^2} \tag{1}$$

at the points t_1, t_2 and t_3 respectively, then the equation of λ_i (for $i = 1, 2, 3$) will be

$$x = \frac{1}{(1 + t_i)(1 + t)}. \tag{2}$$

The intersection of λ_1 and λ_2 is given by the equation

$$x_{12} = \frac{1}{(1 + t_1)(1 + t_2)}, \tag{3}$$

for if we let $t = t_2$ in the equation of λ_1, it is identical with (3); therefore (3) is the equation of a point on line λ_1; and similarly, if we let $t = t_1$, in the equation of λ_2, this equation becomes the same as (3); therefore (3) is the equation of a point on the line λ_2, consequently (3) is the equation of the intersection of the two lines λ_1 and λ_2. In like manner, we obtain the intersections of λ_2 and λ_3, and λ_3 and λ_1. Its conjugate equation is

$$\bar{x}_{12} = \frac{t_1 t_2}{(1 + t_1)(1 + t_2)}. \tag{4}$$

We shall now proceed to reflect these intersections x_{12}, x_{23}, x_{31} in an arbitrary line, say the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{5}$$

From section 1 we see that reflexion in the line is given by the equation

$$\frac{x}{a} + \frac{\bar{y}}{\bar{a}} = 1. \tag{6}$$

Therefore, reflecting the point x_{12} (the intersection of λ_1 and λ_2) whose coördinates are

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)}; \qquad \bar{x} = \frac{t_1 t_2}{(1+t_1)(1+t_2)}, \tag{7}$$

we obtain, as its reflexion in the line, the point

$$\frac{x}{a} = 1 - \frac{t_1 t_2}{\bar{a}(1+t_1)(1+t_2)}, \tag{8}$$

its conjugate

$$\frac{\bar{x}}{\bar{a}} = 1 - \frac{1}{a(1+t_1)(1+t_2)}. \tag{9}$$

In a like manner we can obtain the reflexions of the points x_{23} and x_{31} in the line.

The equation of a tangent to the parabola at the point t_1 is

$$x = \frac{1}{(1+t_1)(1+t)} \tag{10}$$

and its conjugate equation is

$$\bar{x} = \frac{t_1 t}{(1+t_1)(1+t)}. \tag{11}$$

Adding (10) and (11) we have

$$x + \bar{x} = \frac{1}{(1+t_1)(1+t)} + \frac{t_1 t}{(1+t_1)(1+t)}, \tag{12}$$

or

$$x + \frac{\bar{x}}{t_1} = \frac{1}{1+t_1}, \tag{13}$$

which is a self-conjugate equation of the tangent to the parabola at the point t_1. A line perpendicular to this tangent will be of the form

$$x - \frac{\bar{x}}{t_1} = c - \frac{\bar{c}}{t_1}. \tag{14}$$

Consequently, the equation of a line on the reflexion of x_{23} and perpendicular to the line λ_1 (which is tangent to the parabola at the point t_1) is

$$x - \frac{\bar{x}}{t_1} = a - \frac{a t_2 t_3}{\bar{a}(1+t_2)(1+t_3)} - \frac{1}{t_1}\left[\bar{a} - \frac{\bar{a}}{a(1+t_2)(1+t_3)}\right]. \tag{15}$$

For convenience let us rewrite (15) in the following manner:

$$t_1x - \bar{x} = at_1 - \frac{at_1t_2t_3(1+t_1)}{\bar{a}(1+t_1)(1+t_2)(1+t_3)} - \bar{a} + \frac{\bar{a}(1+t_1)}{a(1+t_1)(1+t_2)(1+t_3)} \tag{16}$$

or

$$t_1x - \bar{x} = at_1 - \frac{as_3(1+t_1)}{\bar{a}\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}(1+t_1)}{a\overset{3}{\pi}}, \tag{17}$$

where $\overset{3}{\pi} = (1+t_1)(1+t_2)(1+t_3)$; $s_3 = t_1t_2t_3$.

Let us replace t_1 in (17) by the parameter t, and we obtain the equation

$$tx - \bar{x} = at - \frac{as_3(1+t)}{\bar{a}\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}(1+t)}{a\overset{3}{\pi}}. \tag{18}$$

Now then, if we let $t = t_2$, it is easily seen that (18) becomes the equation of a line on the reflexion of x_{31} and perpendicular to the line λ_2; similarly when $t = t_3$, (18) represents the line on the reflexion of x_{12} and perpendicular to the line λ_3.

Thus, by means of a process called "interpolation" we are enabled to write one equation, which as the parameter "t" assumes the values t_1, t_2, t_3 picks up the three lines μ_1, μ_2 and μ_3. Therefore, the three lines μ_1, μ_2 and μ_3 must intersect in a point, and their point of intersection is expressed by the equation

$$x_{123} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}}. \tag{19}$$

Hence we have the theorem of Desboves:

Given three lines and an extra line; if we reflect the vertices of the three lines in the arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, the three perpendiculars will meet in a point.*

We will call this point the associated point of the line λ with reference to the three given lines λ_1, λ_2, λ_3.

1—*b*. Again, let us consider four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1+t)^2}. \tag{1}$$

We will recall that λ_i (for $i = 1, 2, 3, 4$) is expressed by the equation

$$x = \frac{1}{(1+t_i)(1+t)}. \tag{2}$$

For many purposes the circumcenter of a 3-line plays the part of the intersection of a 2-line. We will now find the circumcenter of three tangents to the parabola. First, let us examine the intersections of the 3-line formed by λ_1, λ_2, λ_3. On page 125 we defined the intersection of two tangents λ_i and λ_j as follows:

$$x_{ij} = \frac{1}{(1 + t_i)(1 + t_j)} . \tag{3}$$

Therefore, the intersections of λ_1, λ_2; λ_2, λ_3 and λ_3, λ_1 have as their respective equations

$$x_{12} = \frac{1}{(1 + t_1)(1 + t_2)}, \tag{4}$$

$$x_{23} = \frac{1}{(1 + t_2)(1 + t_3)}, \tag{5}$$

$$x_{31} = \frac{1}{(1 + t_3)(1 + t_1)} . \tag{6}$$

By means of interpolation we are enabled to write an equation that will pick up all three of these points, such an equation is

$$x = \frac{(1 + t)}{(1 + t_1)(1 + t_2)(1 + t_3)}, \tag{7}$$

for when $t = t_1$, (7) reduces to (5), and therefore (7) passes through the intersection of λ_2 and λ_3, that is, (7) is on the point x_{23}; when $t = t_2$, (7) reduces to (6), and is therefore on x_{31}; and finally when $t = t_3$, (7) reduces to (4) and passes through the point x_{12}. Consequently (7) represents a curve which passes through the three points x_{12}, x_{23} and x_{31}. (7) is the map-equation of a circle, whose center is

$$x_{123} = \frac{1}{(1 + t_1)(1 + t_2)(1 + t_3)} . \tag{8}$$

and the conjugate equation is

$$\bar{x}_{123} = \frac{s_3}{(1 + t_1)(1 + t_2)(1 + t_3)} . \tag{9}$$

This is then the circumcenter. Therefore, associated with four undirected lines, we will have four circumcenters, one for each 3-line.

We shall now proceed as we did in section 1—*a*, page 125; first we will reflect the circumcenter x_{123} in an arbitrary line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{10}$$

and then we will erect a perpendicular to λ_4 on this reflexion. Since reflexion in a line is given by

$$\frac{x}{a}+\frac{\bar{y}}{\bar{a}}=1 \tag{11}$$

we will obtain as the reflexion of x_{123} in the line (10)

$$\frac{x}{a}=1-\frac{1}{\bar{a}}\left[\frac{s_3}{(1+t_1)(1+t_2)(1+t_3)}\right] \tag{12}$$

and its conjugate equation

$$\frac{\bar{x}}{\bar{a}}=1-\frac{1}{a}\left[\frac{1}{(1+t_1)(1+t_2)(1+t_3)}\right]. \tag{13}$$

Now since λ_4 is a tangent to the parabola

$$x=\frac{1}{(1+t)^2} \tag{14}$$

at the point t_4, we can write its equation as

$$x+\frac{\bar{x}}{t_4}=\frac{1}{1+t_4}. \tag{15}$$

A line $\perp$ to (15) will be of the form

$$x-\frac{\bar{x}}{t_4}=c-\frac{\bar{c}}{t_4}. \tag{16}$$

Therefore, the equation of a line perpendicular to λ_4, and on the reflexion of the point x_{123}, in the arbitrary line

$$\frac{x}{a}+\frac{\bar{x}}{\bar{a}}=1, \tag{17}$$

will have the following equation, namely,

$$x-\frac{\bar{x}}{t_4}=a-\frac{as_3'}{\bar{a}\overset{3}{\pi}}-\frac{\bar{a}}{t_4}+\frac{\bar{a}}{a\overset{3}{\pi}t_4} \tag{18}$$

or

$$t_4x-\bar{x}=at_4-\frac{as_4}{\bar{a}\overset{3}{\pi}}-\bar{a}+\frac{\bar{a}}{a\overset{3}{\pi}}, \tag{19}$$

where $\overset{3}{\pi}=(1+t_1)(1+t_2)(1+t_3)$; $s_4=t_1t_2t_3t_4$ and $s_3'=t_1t_2t_3$. Similarly, the equation of a line perpendicular to λ_3, and on the reflexion of the point x_{124}, in the arbitrary line, is

$$x-\frac{\bar{x}}{t_3}=a-\frac{as_3'}{\bar{a}\overset{3}{\pi}}-\frac{\bar{a}}{t_3}+\frac{\bar{a}}{a\overset{3}{\pi}t_3} \tag{20}$$

or

$$t_3x - \bar{x} = at_3 - \frac{as_4}{a\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}}{a\overset{3}{\pi}}, \tag{21}$$

where $\overset{3}{\pi} = (1 + t_1)(1 + t_2)(1 + t_4)$; $s_4 = t_1t_2t_3t_4$ and $s_3' = t_1t_2t_4$.

Again, by the process called "interpolation," we are enabled to write an equation that will pick up these four lines, namely,

$$tx - \bar{x} = at - \frac{as_4(1 + t)}{\bar{a}\overset{4}{\pi}} - \bar{a} + \frac{\bar{a}(1 + t)}{a\overset{4}{\pi}} \tag{22}$$

(where $\overset{4}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)$; $s_4 = t_1t_2t_3t_4$), for when $t = t_4$, (22) reduces to (19), and similarly when $t = t_3$, (22) reduces to (21), etc. Thus, it is easily seen that (22) picks up the four lines which are perpendicular to λ_1, λ_2, λ_3 and λ_4 respectively, and which also lie, respectively, on the reflexions of x_{234}, x_{134}, x_{124} and x_{123} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{23}$$

Moreover, (22) gives the intersection of these four perpendicular lines. Consequently, the point of intersection of the four lines is

$$x_{1234} = a - \frac{as_4}{\bar{a}\overset{4}{\pi}} + \frac{\bar{a}}{a\overset{4}{\pi}} \tag{24}$$

and its conjugate equation is

$$\bar{x}_{1234} = \bar{a} - \frac{\bar{a}}{a\overset{4}{\pi}} + \frac{as_4}{\bar{a}\overset{4}{\pi}},$$

where $\overset{4}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)$; $s_4 = t_1t_2t_3t_4$.

Hence we can state the following theorem:

Given four lines and an extra line, if we reflect the circumcenters of the four 3-lines in the arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, the four perpendiculars will meet in a point.*

1—*c*. Now, we shall consider five undirected lines as the tangents of a double parabola. The equation of the double parabola is

$$x = \frac{A}{(\alpha - t)^2} + \frac{B}{(\beta - t)^2}, \tag{1}$$

provided the cusp condition $dx/dt = 0$; for t a turn. In this case the cusp condition is $A\alpha^3 = A$; $B\beta^3 = B$.

Let the five lines under consideration be designated as λ_1, λ_2, λ_3, λ_4 and λ_5. The equation of a tangent to the double parabola at the point t_1 is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \tag{2}$$

or, in general, the equation of λ_i $(i = 1, 2, 3, 4, 5)$ is

$$x = \frac{A}{(\alpha - t_i)(\alpha - t)} + \frac{B}{(\beta - t_i)(\beta - t)}. \tag{3}$$

We shall now proceed to find the intersection of two tangents, say λ_1 and λ_2. Since the equation of λ_1 is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \tag{4}$$

and that of λ_2 is

$$x = \frac{A}{(\alpha - t_2)(\alpha - t)} + \frac{B}{(\beta - t_2)(\beta - t)}, \tag{5}$$

then the equation of x_{12}, the intersection of λ_1, λ_2, will be

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)}. \tag{6}$$

Similarly, we can find the intersections of x_{13}, x_{23}, x_{14}, x_{15}, etc.

In section 1—*b*, we found that to each 3-line there was associated a circumcenter; therefore to every 3-line arising from these five lines there is associated a circumcenter. Since we know the equation of the intersection of λ_1, λ_2, we can, by symmetry, write the equations of the intersections of λ_1, λ_3 and λ_2, λ_3. Thus

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)} \tag{7}$$

$$x_{23} = \frac{A}{(\alpha - t_2)(\alpha - t_3)} + \frac{B}{(\beta - t_2)(\beta - t_3)}, \tag{8}$$

$$x_{31} = \frac{A}{(\alpha - t_3)(\alpha - t_1)} + \frac{B}{(\beta - t_3)(\beta - t_1)}. \tag{9}$$

By means of interpolation we can get the equation of the circumcircle, which passes through the points x_{12}, x_{23}, x_{31}, for if $t = t_i$ (where $i = 1, 2, 3$) in the equation

$$x = \frac{A(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)}, \tag{10}$$

(10) becomes successively (8), (9) and (7).

The center of the circle (10) is

$$x_{123} = \frac{A\alpha}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B\beta}{(\beta - t_1)(\beta - t_2)(\beta - t_3)}. \tag{11}$$

But from the four lines we can form four 3-lines, and since for every 3-line there is a circumcenter, therefore, from the four lines λ_1, λ_2, λ_3 and λ_4 we can obtain the four circumcenters x_{123}, x_{124}, x_{134}, x_{234}. From (11) we can write their equations, by means of symmetry.

Interpolating, we obtain

$$x = \frac{A\alpha(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)}, \quad (12)$$

which equation, obviously, as t assumes successively the values t_1, t_2, t_3, t_4, picks up the points x_{123}, x_{124}, x_{134}, x_{234}.

But (12) is the equation of a circle, and its center is

$$x_{1234} = \frac{A\alpha^2}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta^2}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)}, \quad (13)$$

$$\bar{x}_{1234} = \frac{\bar{A}(1/\alpha^2)}{(1/\alpha - 1/t_1)(1/\alpha - 1/t_2)(1/\alpha - 1/t_3)(1/\alpha - 1/t_4)} + \frac{\bar{B}(1/\beta^2)}{(1/\beta - 1/t_1)(1/\beta - 1/t_2)(1/\beta - 1/t_3)(1/\beta - 1/t_4)}. \quad (14)$$

Applying the cusp condition $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$ and letting

$$P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)$$

and

$$Q_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4),$$

we have

$$\bar{x} = \frac{\bar{A}\alpha^2 s_4}{P_4} + \frac{\bar{B}\beta^2 s_4}{Q_4} \quad (15)$$

$$= \frac{As_4}{\alpha P_4} + \frac{Bs_4}{\beta Q_4} \quad (16)$$

$$= \sum^{2} \frac{As_4}{\alpha P_4}, \quad (17)$$

where $s_4 = t_1 t_2 t_3 t_4$.

If now we reflect this point x_{1234} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1, \quad (18)$$

we obtain, as its reflexion, the point

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{A s_4}{\alpha P_4}, \tag{19}$$

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_4}. \tag{20}$$

The equation of a tangent to the double parabola at the point t_5 is

$$x = \frac{A}{(\alpha - t_5)(\alpha - t)} + \frac{B}{(\beta - t_5)(\beta - t)} \tag{21}$$

and its conjugate equation is

$$\bar{x} = \frac{\bar{A}\alpha^2 t t_5}{(\alpha - t_5)(\alpha - t)} + \frac{\bar{B}\beta^2 t t_5}{(\beta - t_5)(\beta - t)}. \tag{22}$$

Since the cusp condition is $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$, we can write (22) as

$$\frac{\bar{x}}{t_5} = \frac{At}{\alpha(\alpha - t_5)(\alpha - t)} + \frac{Bt}{\beta(\beta - t_5)(\beta - t)}. \tag{23}$$

Subtracting (23) from (21) we obtain a self-conjugate expression for the tangent to the double parabola at the point t_5, namely,

$$x - \frac{\bar{x}}{t_5} = \frac{A}{\alpha(\alpha - t_5)} + \frac{B}{\beta(\beta - t_5)} \tag{24}$$

or

$$x - \frac{\bar{x}}{t_5} = \sum^{2} \frac{A}{\alpha(\alpha - t_5)}. \tag{25}$$

A line perpendicular to this tangent and on the point $\bar{x}_{1234}$ is expressed by the equation

$$x + \frac{\bar{x}}{t_5} = a - \frac{a}{\bar{a}} \sum^{2} \frac{A s_4}{\alpha P_4} + \frac{\bar{a}}{t_5} - \frac{\bar{a}}{a t_5} \sum^{2} \frac{A\alpha^2}{P_4} \tag{26}$$

or

$$t_5 x + \bar{x} = a t_5 - \frac{a}{\bar{a}} \sum^{2} \frac{A s_5}{\alpha P_4} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_4}, \tag{27}$$

where

$$P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4),$$
$$Q_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)$$

and $S_5 = t_1 t_2 t_3 t_4 t_5$. But in this case we will have five perpendicular lines, namely, l_1 perpendicular to λ_1 and on $\bar{x}_{2345}$; $l_2 \perp \lambda_2$ and on $\bar{x}_{1345}$; $l_3 \perp \lambda_3$ and on $\bar{x}_{1245}$; $l_4 \perp \lambda_4$ and on $\bar{x}_{1235}$, and $l_5 \perp \lambda_5$ and on $\bar{x}_{1234}$.

Symmetrizing so as to pick up these five lines, we obtain the equation

$$tx + \bar{x} = at - \frac{a}{\bar{a}} \sum^{2} \frac{As_5(\alpha - t)}{\alpha P_5} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{P_5} . \qquad (28)$$

It is at once evident, when $t = t_i$ (where $i = 1, 2, 3, 4, 5$), (28) gives us the line l_i ($i = 1, 2, 3, 4, 5$).

But (28) is the equation of the intersection of the five lines l_1, l_2, l_3, l_4 and l_5. This point has the equation

$$x_{12345} = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_5}{P_5} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_5} . \qquad (29)$$

Consequently, we have the theorem:

Given five lines and an extra line; if we reflect the centric points of the five 4-lines in an arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, the five perpendiculars will meet in a point.*

In general, we can say:

Given n lines and an extra line; if we reflect the centric points of the n "$(n-1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the n perpendiculars will meet in a point.

SECOND CHAIN OF THEOREMS.

2—*a*. Given four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1+t)^2} \qquad (1)$$

at the points t_1, t_2, t_3 and t_4 respectively, then the equation of λ_i ($i = 1, 2, 3, 4$) is

$$x = \frac{1}{(1+t_i)(1+t)} . \qquad (2)$$

With four undirected lines we can form four 3-lines for $C_3^4 = 4$. Here, the four 3-lines are composed of $\lambda_1, \lambda_2, \lambda_3$; $\lambda_1, \lambda_2, \lambda_4$; $\lambda_1, \lambda_3, \lambda_4$; and $\lambda_2, \lambda_3, \lambda_4$ respectively. Now, by the theorem of section 1—*a* we found that to every 3-line $\lambda_i, \lambda_j, \lambda_k$ there is associated a point x_{ijk}. Consequently, to the 3-line $\lambda_1, \lambda_2, \lambda_3$ is associated a point x_{123}; to the 3-line $\lambda_1, \lambda_2, \lambda_4$ the point x_{124}, and so forth.

The equation of the point x_{123} is

$$x_{123} = a - \frac{as_3}{\bar{a}\,\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} , \qquad (3)$$

where $\overset{3}{\pi} = (1+t_1)(1+t_2)(1+t_3)$; $s_3 = t_1t_2t_3$ and its conjugate equation is

$$\bar{x}_{123} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}. \tag{4}$$

Similarly, the equation of the point x_{124} is

$$x_{124} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} \tag{5}$$

and

$$\bar{x}_{124} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}, \tag{6}$$

where $\overset{3}{\pi} = (1+t_1)(1+t_2)(1+t_4)$; $s_3 = t_1t_2t_4$. And so, in general, we have, as the equation of x_{ijk}, the point associated with the 3-line $\lambda_i, \lambda_j, \lambda_k$:

$$x_{ijk} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} \tag{7}$$

and

$$\bar{x}_{ijk} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}, \tag{8}$$

where $\overset{3}{\pi} = (1+t_i)(1+t_j)(1+t_k)$; $s_3 = t_it_jt_k$.

Symmetrizing so as to pick up the four points x_{123}, x_{124}, x_{134}, x_{234}, we obtain the equation

$$x = a - \frac{as_4(1+t)}{\bar{a}t\overset{4}{\pi}} + \frac{\bar{a}(1+t)}{a\overset{4}{\pi}} \tag{9}$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}(1+t)}{a\overset{4}{\pi}} + \frac{as_4(1+t)}{\bar{a}t\overset{4}{\pi}}, \tag{10}$$

where $\overset{4}{\pi} = (1+t_1)(1+t_2)(1+t_3)(1+t_4)$; $s_4 = t_1t_2t_3t_4$. Adding (9) and (10) we have

$$x + \bar{x} = a + \bar{a},$$

which is the equation of a vertical line.

Consequently, we have another theorem, namely:

Given four lines and an extra line λ, *the four associated points, which arise from the four 3-lines and an arbitrary line* λ, *lie on a line.*

Cwojdzinski states this theorem.*

2—*b*. Given five lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ as tangents to the double parabola

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t)^2} \tag{1}$$

* *Archiv der Mathematik und Physik*, Vol. 1, 1901, p. 180.

(cusp condition being $\bar{A}_i\alpha_i^3 = A_i$), at the points t_1, t_2, t_3, t_4 and t_5 respectively, then the equation of the line λ_i ($i = 1, 2, 3, 4, 5$) is

$$x = \sum^{2} \frac{A_i}{(\bar{\alpha}_i - t_i)(\alpha_i - t)} \cdot \tag{2}$$

With five undirected lines we can form five 4-lines for $C_4^5 = 5$. Here the five 4-lines are composed of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; $\lambda_1, \lambda_3, \lambda_4, \lambda_5$; $\lambda_1, \lambda_2, \lambda_4, \lambda_5$; $\lambda_1, \lambda_2, \lambda_3, \lambda_5$; $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively. Now, by theorem 1—*b* we found that to every 4-line $\lambda_i, \lambda_j, \lambda_k, \lambda_l$ there is associated a point x_{ijkl}. Consequently, to the above-named 4-lines are associated the points x_{1234}, x_{1345}, x_{1235}, x_{1245}, x_{2345}. The equation of an associated point given in section 1—*b* was for four lines taken as tangents to a parabola with the arbitrary line λ; here it will be necessary to find the equation of the associated point, the four lines being taken as tangents to a double parabola.

From 1—*c* we have the equation of the circumcenter for three lines $\lambda_1, \lambda_2, \lambda_3$ taken as tangents to a double parabola, namely,

$$x_{123} = \sum^{2} \frac{A\alpha}{\overset{3}{\pi}}, \tag{3}$$

and its conjugate is

$$\bar{x}_{123} = -\sum^{2} \frac{As_3}{\alpha\overset{3}{\pi}} \cdot \tag{4}$$

Reflecting this point in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{5}$$

we obtain the point

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_3}{\alpha\overset{3}{\pi}}, \tag{6}$$

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha}{\overset{3}{\pi}} \cdot \tag{7}$$

A line $\perp$ to the tangent λ_4 and on the reflexion of the point x_{123} in the arbitrary line λ is

$$t_4 x + \bar{x} = t_4 a + \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha\overset{3}{\pi}} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha}{\overset{3}{\pi}} \cdot \tag{8}$$

Now interpolating so as to pick up these four perpendiculars, we obtain the equation

$$tx + \bar{x} = ta + \frac{a}{\bar{a}} \sum^{2} \frac{As_4(\alpha - t)}{\alpha\overset{4}{\pi}} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha(\alpha - t)}{\overset{4}{\pi}}, \tag{9}$$

which is an associated point

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{A s_4}{\alpha \overset{4}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha}{\overset{4}{\pi}}, \tag{10}$$

where $\overset{4}{\pi} = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)$; $s_4 = t_1 t_2 t_3 t_4$.

Interpolating so as to pick up these five associated points, we have

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{A s_5 (\alpha - t)}{\alpha t \overset{5}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha(\alpha - t)}{\overset{5}{\pi}}. \tag{11}$$

This is the equation of an ellipse, and its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{\overset{5}{\pi}} + \frac{a}{\bar{a}} \sum^{2} \frac{A s_5 (\alpha - t)}{t \overset{5}{\pi}}, \tag{12}$$

where $\overset{5}{\pi} = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)(\alpha_i - t_5)$; $s_5 = t_1 t_2 t_3 t_4 t_5$.

Hence we have the theorem:

Given five lines and an extra line λ, *the five associated points, which arise from the five* 4*-lines, and an arbitrary line* λ, *lie on an ellipse.*

2—*c*. Given six lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ as tangents to the double parabola

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t)^2} \tag{1}$$

(cusp condition being $\overline{A}_i \alpha_i^3 = A$), at the points $t_1, t_2, t_3, t_4, t_5, t_6$ respectively, then the equation of the tangent λ_i $(i = 1, 2, 3, 4, 5, 6)$ is

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t_i)(\alpha_i - t)}. \tag{2}$$

With six undirected lines we can form six 5-lines for $C_5^6 = 6$. From theorem 1—*c* we found that to every 5-line and an arbitrary line there is associated a point. Therefore to six lines and an arbitrary line there will be associated six points. The equation of the point x_{12345} is

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{A s_5}{\alpha \overset{5}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{\overset{5}{\pi}}. \tag{3}$$

Symmetrizing so as to pick up the six points, namely, x_{12345}, x_{12346}, x_{12456}, x_{13456}, x_{12356}, x_{23456}, we have

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{A s_6 (\alpha - t)}{\alpha t \overset{6}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{\overset{6}{\pi}}, \tag{4}$$

which is the map-equation of an ellipse. Its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\sum^{2}\frac{A\alpha^3(\alpha - t)}{\pi^6} - \frac{a}{\bar{a}}\sum^{2}\frac{As_6(\alpha - t)}{t\pi^6}. \tag{5}$$

Hence the theorem:

Given six lines and an extra line λ*; the six associated points, which arise from the six* 5*-lines and an arbitrary line* λ*, lie on an ellipse.*

2—*g*. In general, we can say:

Given n lines and an arbitrary line λ; the n associated points, which arise from the n "$(n-1)$-lines" and an arbitrary line λ, lie on an ellipse.

Directed Lines.

FIRST CHAIN OF THEOREMS.

I—*a*. Suppose we have four directed lines L_1, L_2, L_3, L_4 given as tangents to a parastroid at the points t_1, t_2, t_3 and t_4; then since the equation of the parastroid is

$$t^4 - t^3x + \mu t^2 - t\bar{x} + 1 = 0 \tag{1}$$

(where μ is real), the equations of L_1, L_2, L_3, L_4 will be respectively

$$t_1^4 - t_1^3x + \mu t_1^2 - t_1\bar{x} + 1 = 0, \tag{2}$$
$$t_2^4 - t_2^3x + \mu t_2^2 - t_2\bar{x} + 1 = 0, \tag{3}$$
$$t_3^4 - t_3^3x + \mu t_3^2 - t_3\bar{x} + 1 = 0, \tag{4}$$
$$t_4^4 - t_4^3x + \mu t_4^2 - t_4\bar{x} + 1 = 0. \tag{5}$$

The incenter* of the lines L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3} \tag{6}$$

and its conjugate is

$$\bar{x}_{123} = 1/t_1 + 1/t_2 + 1/t_3 + t_1t_2t_3. \tag{7}$$

The reflexion of the point x_{123} in the line λ, namely,

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1, \tag{8}$$

is

$$x = a - \frac{a}{\bar{a}}[1/t_1 + 1/t_2 + 1/t_3 + t_1t_2t_3] \tag{9}$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\left[t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right]. \tag{10}$$

We now desire to drop a perpendicular from the reflexion of x_{123} to the

* Hodgson, Joseph E., "Orthocentric Properties of the Plane Directed n-Line," *Transactions of the American Mathematical Society*, Vol. 13, 1912, p. 199.

line L_4; since a perpendicular to L_4 is of the form

$$t_4^2 x - \bar{x} = t_4^2 c - \bar{c}, \tag{11}$$

in order that this perpendicular pass through the reflexion of x_{123} in the line λ, (11) becomes

$$t_4^2 x - \bar{x} = t_4^2 \left[a - \frac{a}{\bar{a}} \left(\frac{s_2}{s_3} + s_3 \right) \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(s_1 + \frac{1}{s_3} \right) \right]. \tag{12}$$

In order to interpolate, so as to be able to pick up the four perpendiculars, namely, the one from the reflexion of x_{234} to L_1; from the reflexion of x_{134} to L_2, etc., we will introduce the following symmetric functions:

$$\begin{aligned}
&s_1 = t_1 + t_2 + t_3, && \sigma_1 = t_1 + t_2 + t_3 + t_4,\\
&s_2 = t_1t_2 + t_1t_3 + t_2t_3, && \sigma_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4,\\
&s_3 = t_1t_2t_3, && \sigma_3 = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4,\\
& && \sigma_4 = t_1t_2t_3t_4;\\
& \therefore && s_1 = \sigma_1 - t_4,\\
& && s_2 = \sigma_2 - t_4\sigma_1 - t_4^2,\\
& && s_3 = \sigma_4/t_4.
\end{aligned}$$

Now then, making use of the above symmetric functions, (12) can be written as

$$t_4^2 x - \bar{x} = t_4^2 a - \frac{a}{\bar{a}} \left[\frac{(t_4^2\sigma_2 - t_4^3\sigma_1 + t_4^4)t_4}{\sigma_4} + \frac{t_4\sigma_4^2}{\sigma_4} \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(s_1 + \frac{1}{s_3} \right) \right] \tag{13}$$

But

$$t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4 = 0, \tag{14}$$

$$\therefore \quad t_4 - \sigma_1 t^3 + \sigma_2 t^2 = \sigma^3 t - \sigma_4. \tag{15}$$

Making use of this relation in (13), we obtain the equation

$$t_4^2 x - \bar{x} = t_4^2 a - \left[\frac{at_4^2\sigma_3 - at_4\sigma_4 + at_4\sigma_4^2}{\bar{a}\sigma_4} \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(s_1 + \frac{1}{s_3} \right) \right]. \tag{16}$$

Thus (16) is the equation of a line which passes through the reflexion of x_{123}, and is perpendicular to L_4; which for convenience we will call R_4. There will be four such lines, namely, R_1, R_2, R_3, R_4, which will be obtained from the original four directed lines L_1, L_2, L_3 and L_4.

Symmetrizing so as to pick up these four lines, we have

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2\sigma_3 - at\sigma_4 + at\sigma_4^2}{\bar{a}\sigma_4} \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(\sigma_1 - t + \frac{t}{\sigma_4} \right) \right], \tag{17}$$

which is a circle.

As a result, we have the theorem:

Given four directed lines and an extra line λ*; if we reflect the incenters of the four 3-lines in an arbitrary line* λ*, and then drop perpendiculars from these reflexions to the remaining line, these four perpendiculars will touch a circle.*

I—*b*. We will next consider the five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a parastroid, at the points t_1, t_2, t_3, t_4 and t_5 respectively. L_i (where $i = 1, 2, 3, 4, 5$) will be expressed by the equation

$$t_i^4 - t_i^3 x + \mu t_i^2 - t_i \bar{x} + 1 = 0, \tag{1}$$

where μ is real.

As given in section I—*a*, the incenter of the 3-line L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}. \tag{2}$$

If now, instead of taking three directed lines, we take four directed lines, namely, L_1, L_2, L_3, L_4, then we will be able to form four 3-lines, and since there is an incenter associated with each 3-line, there will be four incenters associated with the four directed lines. Interpolating, so as to pick up these four incenters, we obtain the equation

$$x = \sigma_1 - t + \frac{t}{\sigma_4}, \tag{3}$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4$ and $\sigma_4 = t_1 t_2 t_3 t_4$. Now (3) is the equation of a circle whose center is

$$x_{1234} = \sigma_1. \tag{4}$$

We will now proceed to reflect this center in an arbitrary line, say,

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{5}$$

Doing this, we obtain as the reflexion of x_{1234}

$$x = a - \frac{a}{\bar{a}}\left[\frac{\sigma_3}{\sigma_4}\right] \tag{6}$$

and its conjugate equation

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}[\sigma_1]. \tag{7}$$

As in the previous cases, we desire to drop a perpendicular from the reflexion of x_{1234} to the remaining line L_5. The equation of such a perpendicular is

$$t_5^2 x - \bar{x} = t_5^2\left[a - \frac{a}{\bar{a}}\left(\frac{\sigma_3}{\sigma_4}\right)\right] - \left[\bar{a} - \frac{\bar{a}}{a}(\sigma_1)\right]. \tag{8}$$

But in this case, since we have five directed lines, we will obtain five perpendiculars similar to the one expressed by equation (8). In order to interpolate we will have to introduce the following symmetric functions:

$$\sigma_1 = t_1 + t_2 + t_3 + t_4, \qquad P_1 = \sum^{5} t_1,$$

$$\sigma_2 = \sum^{6} t_1 t_2, \qquad P_2 = \sum^{10} t_1 t_2,$$

$$\sigma_3 = \sum^{4} t_1 t_2 t_3, \qquad P_3 = \sum^{10} t_1 t_2 t_3,$$

$$\sigma_4 = t_1 t_2 t_3 t_4, \qquad P_4 = \sum^{5} t_1 t_2 t_3 t_4,$$

$$P_5 = t_1 t_2 t_3 t_4 t_5;$$

$$\therefore \quad \sigma_1 = P_1 - t_5,$$
$$\sigma_2 = P_2 - P_1 t_5 + t_5^2,$$
$$\sigma_3 = P_3 - t_5 \sigma_2,$$
$$\sigma_4 = P_5 / t_5.$$

But since the "Ps" can be considered as the symmetric functions of a quintic in t, we obtain the relation

$$t^5 - P_1 t^4 + P_2 t^3 - P_3 t^2 + P_4 t - P_5 = 0, \tag{9}$$

from which we have

$$t_5^2 \sigma_3 = t_5^2 P_3 - t_5^3 \sigma_2 = t_5^2 P_3 - t_5^3 P_2 + t_5^4 P_1 - t_5^5 \tag{10}$$

and also that

$$t_5^2 P_3 - t_5^3 P_2 + t_5^4 P_1 - t_5^5 = t_5 P_4 - P_5. \tag{11}$$

Making use of the relations that exist among these two sets of symmetric functions, (8) can be written as

$$t_5^2 x - \bar{x} = t_5^2 a - \frac{a}{\bar{a}} \left[\frac{(t_5^2 P_3 - t_5^3 \sigma_2) t_5}{P_5} \right] - \left[\bar{a} - \frac{\bar{a}}{a} (\sigma_1) \right] \tag{12}$$

or applying (11) to (12) we obtain

$$t_5^2 x - \bar{x} = t_5^2 a - \frac{a}{\bar{a}} \left[\frac{P_4 t_5^2 - P_5 t_5}{P_5} \right] - \left[\bar{a} - \frac{\bar{a}}{a} (\sigma_1) \right]. \tag{13}$$

Interpolating so as to pick up the five perpendiculars, we have the equation

$$t^2 x - \bar{x} = t^2 a - \frac{a}{\bar{a}} \left[\frac{t^2 P_4 - t P_5}{P_5} \right] - \left[\bar{a} - \frac{\bar{a}}{a} (P_1 - t) \right], \tag{14}$$

which is a circle.

Consequently, we have the following theorem:

Given five directed lines and an extra line; if we reflect the centric points of the five 4-lines in an arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, these five perpendiculars will touch a circle.*

I—*g*. In general, we can say:

Given n directed lines and an extra line; if we reflect the centric points of the n "$(n-1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, these n perpendiculars will touch a circle.

SECOND CHAIN OF THEOREMS.

II—*a*. Let us consider five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a parastroid, at the points t_1, t_2, t_3, t_4, t_5 respectively; then L_i (where $i = 1, 2, 3, 4, 5$) can be expressed as

$$t_i^4 - t_i^3 x + t_i^2 \mu - t_i \bar{x} + 1 = 0, \tag{1}$$

where μ is real.

By the previous theorem I—*a* we found that to every four directed lines and an extra line λ there is associated a circle, and since there are five directed lines, we will have five 4-lines and, therefore, five circles associated with the five directed lines. The equation of one of these circles is

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2\sigma_3 - at\sigma_4 + at\sigma_4^2}{\bar{a}\sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(\sigma_1 - t + \frac{t}{\sigma_4}\right)\right]. \tag{2}$$

Symmetrizing so as to pick up the five circles, we obtain

$$\begin{aligned} t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}+\frac{1}{t_5}-\frac{1}{T}\right)\right] + \frac{at}{\bar{a}} - \frac{at\sigma_5}{\bar{a}T} \\ - \bar{a} + \frac{\bar{a}}{a}\left[\sigma_1 - t - T + \frac{tT}{\sigma_5}\right], \end{aligned} \tag{3}$$

where

$$\begin{aligned} \sigma_1 &= t_1 + t_2 + t_3 + t_4 + t_5; \\ \sigma_5 &= t_1 t_2 t_3 t_4 t_5. \end{aligned}$$

For convenience, let us denote σ_5 by T, then (3) becomes

$$\begin{aligned} t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{\sigma_4}{T}-\frac{1}{T}\right)\right] + \frac{at}{\bar{a}} - \frac{atT}{\bar{a}T} - \bar{a} \\ + \frac{\bar{a}}{a}\left[\sigma_1 - t - T - \frac{tT}{T}\right] \end{aligned} \tag{4}$$

or

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{\sigma_4}{T}-\frac{1}{T}\right)\right] - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - T]. \tag{5}$$

Now then let $T = t$, and (5) becomes

$$t^2x - \bar{x} = t^2a - \left[\frac{at(\sigma_4 - 1)}{\bar{a}}\right] - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - t] \tag{6}$$

or

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - a^2t(\sigma_4 - 1) - \bar{a}^2a + \bar{a}^2(\sigma_1 - t), \tag{7}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - a^2t\sigma_4 + a^2t - \bar{a}^2t + \bar{a}^2\sigma_1 - \bar{a}^2a, \tag{8}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - t(a^2\sigma_4 - a^2 + \bar{a}^2) + \bar{a}^2(\sigma_1 - a), \tag{9}$$

$$t^2x - \bar{x} = at^2 - \frac{t(a^2\sigma_4 - a^2 + \bar{a}^2)}{a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{a}. \tag{10}$$

But $t = T = \sigma_5$,

$$\therefore \quad x - \frac{\bar{x}}{\sigma_5^2} = a - \frac{(a^2\sigma_4 - a^2 + \bar{a}^2)}{\sigma_5 a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{\sigma_5^2 a}; \tag{11}$$

this is a self-conjugate equation, and is therefore the equation of a line, the line which touches the five circles arising from the five 4-lines.

Hence we have the theorem:

Given five directed lines and an extra line; the five circles, which arise from the five 4-lines and an arbitrary line λ, *all touch the same line.*

II—*b*. Let us consider six directed lines $L_1, L_2, L_3, L_4, L_5, L_6$ as tangents to a cyclogen, at the points $t_1, t_2, t_3, t_4, t_5, t_6$ respectively; then L_i ($i = 1, 2, 3, 4, 5, 6$) can be expressed as

$$t_i^6 - ct_i^5 + xt_i^4 - \mu t_i^3 + \bar{x}t_i^2 - \bar{c}t_i + 1 = 0, \tag{1}$$

where $i = 1, 2, 3, 4, 5, 6$ and μ is real.

By a previous theorem I—*b* we found out that to every five directed lines and an arbitrary line λ there is associated a circle, and since there are six 5-lines which arise from six directed lines, there will be six circles associated with six directed lines.

The equation of one of these circles is

$$t^2x - \bar{x} = t^2a - \frac{a}{\bar{a}}\left[\frac{t^2P_4 - tP_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(P_1 - t)\right]. \tag{2}$$

Symmetrizing so as to pick up these six circles, we obtain the equation

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5} + \frac{1}{t_6} - \frac{1}{T}\right] + \frac{at}{\bar{a}} - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - t - T], \tag{3}$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$; $\sigma_6 = t_1t_2t_3t_4t_5t_6$. For convenience,

let us denote σ_6 by T, then (3) becomes

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{at}{\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a}t}{a} - \frac{\bar{a}T}{a}, \tag{4}$$

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{a^2t}{a\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a}^2t}{a\bar{a}} - \frac{\bar{a}T}{a}. \tag{5}$$

Now then, let $T = t$, and (5) becomes

$$t^2x - \bar{x} = t^2a - \left[\frac{at}{\bar{a}}(\sigma_5 - 1)\right] + \frac{a^2t}{a\bar{a}} - \frac{\bar{a}^2t}{a\bar{a}} - \bar{a} + \frac{\bar{a}}{a}(\sigma_1 - t), \tag{6}$$

$$a\bar{a}(t^2x - \bar{x}) = t^2a^2\bar{a} - a^2t(\sigma_5 - 1) + a^2t - \bar{a}^2t - a\bar{a}^2 + \bar{a}^2(\sigma_1 - 1), \tag{7}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 + t(2a^2 - 2\bar{a}^2 - a^2\sigma_5) + \bar{a}^2(\sigma_1 - a), \tag{8}$$

$$t^2x - \bar{x} = at^2 + \frac{t(2a^2 - 2\bar{a}^2 - a^2\sigma_5)}{a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{a}. \tag{9}$$

But $t = T = \sigma_6$,

$$\therefore\ x - \frac{\bar{x}}{\sigma_6^2} = a + \frac{(2a^2 - 2\bar{a}^2 - a^2\sigma_5)}{\sigma_6 a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{\sigma_6^2 a}; \tag{10}$$

this is a self-conjugate equation, and is therefore the equation of a line, the line which touches the six circles arising from the six 5-lines.

Consequently we have the theorem:

Given six directed lines and an extra line; the six circles, which arise from the six 5-lines and an arbitrary line λ, *all touch the same line.*

And, in general, we can say:

II—c. Given n directed lines and an extra line; the n circles, which arise from the n "(n − 1)-lines" and an arbitrary line λ, *all touch the same line.*

VITA

Flora Dobler Sutton was born in Baltimore, Maryland, June 1890. She was educated in the public schools of Baltimore, and after graduating from the Western High School entered Goucher College and graduated in 1912. In 1915, she entered the Johns Hopkins University as a graduate student and pursued courses in mathematics, education and statistics. In January 1920, she was enrolled as a special student in the School of Hygiene and Public Health, Johns Hopkins University.

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